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FINITE-DEFORMATION CRACK IN AN INFINITE BODY UNDER ANTI-PLANE SIMPLE SHEAR†

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Abstract—The anti-plane shear crack problem in an infinite body is studied within the context of finite-deformation elastostatics. The governing equations are found to be identical to those of certain previously studied small strain plastic problems under anti-plane shear. For a particular strain energy function, an exact solution of the problem is obtained through use of results in the literature.

INTRODUCTION

Most solutions of crack problems are based on small strain theories of elasticity or plasticity. Recently J. K. Knowles and E. Sternberg[1] considered a plane strain crack in a compressible, elastic material under uniform tension at infinity. By taking finite deformation into account, they have derived an asymptotic solution of the singular field near the crack-tip. However, they are able to compute the amplitude of the singular field only for sufficiently low load levels such that the nonlinear behavior is essentially confined to a region small compared to the crack length. In the present paper, the simpler problem of the anti-plane simple shear crack in an incompressible material is considered within the context of finite-deformation elasticity. It is found that, for a particular form of the strain energy function, the problem is reduced to a form formally identical to a problem solved by J. Amazigo[2], and, thereby, an exact solution is obtained. Furthermore, the amplitude of the singular field near the crack-tip is rendered completely determinate for arbitrary large deformations.

FORMULATION OF THE PROBLEM

Let x_i and y_i be the Cartesian coordinates of a typical point in the undeformed and deformed body respectively and let u_i be the displacement components. By symmetry, the problem with the infinite body is equivalent to the problem of a semi-infinite body with an edge crack in the region $\mathscr{D}$ (see Fig. 1).

In the absence of the crack, every point in this infinite body undergoes a simple shearing deformation so that

$$y_3 = x_3 + \gamma_\infty x_2, \qquad y_\alpha = x_\alpha, \qquad \alpha = 1,2 \tag{1}$$

where γ_∞ is a constant. With the introduction of a crack, u_3 becomes a function of x_1 and x_2,

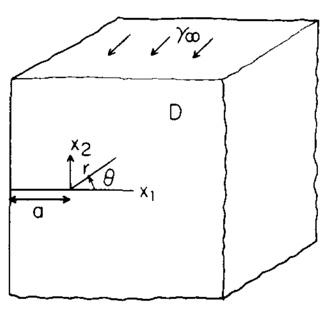

Fig. 1. Geometry.

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and it is still assumed to be the only nonzero displacement component. So

$$y_3 = x_3 + u_3(x_1, x_2), \qquad y_\alpha = x_\alpha. \tag{2}$$

The conditions at infinity remain the same as before, i.e.

$$u_{3,2} = \gamma_\infty, \qquad u_{3,\alpha} = 0. \tag{3}$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress field in the deformed body and define the displacement gradient $\mathbf{F}$, the Jacobian determinant $\mathscr{J}$ of the transformation $\mathbf{y}(\mathbf{x})$ and the Piola stress field $\boldsymbol{\tau}$ as

$$\mathbf{F} = (F_{ij}) = \left(\frac{\partial y_i}{\partial x_j}\right), \qquad \mathscr{J} = \det\left(\frac{\partial y_i}{\partial x_j}\right), \qquad \boldsymbol{\tau} = \mathscr{J}\boldsymbol{\sigma}(\mathbf{F}^{-1})^T \tag{4}$$

where the superscript T denotes transpose of the matrix. The traction-free boundary conditions, from [1], can be prescribed over the undeformed body in terms of the Piola stress tensor τ_{ij} referred to the original Cartesian coordinates, i.e.

$$\tau_{13} = 0 \text{ on } x_1 = -a, \quad -\infty < x_2 < \infty \tag{5a}$$

$$\tau_{23} = 0 \text{ on } -a \le x_1 \le 0, \quad x_2 = 0\pm. \tag{5b}$$

For a homogeneous, isotropic, incompressible material, the strain energy function W is a function of the two principal invariants only, i.e.,

$$W = W(I_1, I_2) \tag{6}$$

where $I_1 = tr(\mathbf{F}^T\mathbf{F})$, $I_2 = \frac{1}{2}[tr(\mathbf{F}^T\mathbf{F})^2 - tr(\mathbf{F}^T\mathbf{F})^2]$. If W is assumed to be a function of I_1 only, the constitutive relation may be written as

$$\boldsymbol{\tau} = \Sigma\mathbf{F} + p(\mathbf{F}^{-1})^T \tag{7}$$

where $\Sigma = 2(\mathrm{d}W/\mathrm{d}I_1)$ and p is the hydrostatic pressure.

It can be shown that [1] the Piola stress field satisfies the following equilibrium equation in the undeformed region

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0 \text{ in } \mathscr{D}. \tag{8}$$

In view of (5a) and (7), (8) gives only one non-trivial equilibrium equation

$$(\Sigma u_{3,1})_{,1} + (\Sigma u_{3,2})_{,2} = 0. \tag{9}$$

Now define

$$\gamma_\alpha = u_{3,\alpha}, \qquad \tau_\alpha = \tau_{3\alpha} = \Sigma u_{3,\alpha} \qquad \alpha = 1,2. \tag{10}$$

It follows immediately from the above and (9) that

$$\gamma_{1,2} - \gamma_{2,1} = 0, \qquad \tau_{1,1} + \tau_{2,2} = 0. \tag{11}$$

If we put

$$\gamma = \sqrt{(\gamma_1{}^2 + \gamma_2{}^2)}, \qquad \tau = \sqrt{(\tau_1{}^2 + \tau_2{}^2)} \tag{12}$$

we find from (10) that τ is a function of γ alone,

$$\tau = \Sigma\gamma = \tau(\gamma).\dagger \tag{13}$$

†Note $\Sigma = \Sigma(I_1) = \Sigma(u_{3,1}^2 + u_{3,2}^2) = \Sigma(\gamma)$.

Equations (10), (11) together with boundary conditions (5) reduce to the same problem considered in [3]. Assume the form of the strain energy function as

$$W = A(I_1 - 3)^n \qquad n > 1/2 \tag{14}$$

where A is a constant and n a parameter. If we now identify n with $\frac{1}{2}(n+1)/n$ and A with $\tau_0(\alpha\gamma_0)^{-1/n}n/(n+1)$ in [2], this problem becomes identical to the problem of the fully plastic crack under anti-plane shear studied by J. Amazigo[2], based on small strain deformation theory. This permits us to obtain an exact global solution to this problem by simply making use of the results in [2]. For $n = 1$, the strain energy function in (14) reduces to that of a neo-Hookean material. As n increases, the material corresponding to (14) becomes more "rubber-elastic". Figure 2 shows the behavior of this class of material in uniaxial tension. Except when $n = 1$, the stress-strain curve based on the strain energy function in (14) is of the pure power type which has infinite (or zero) initial slope in the unstrained state. Consequently, this idealized stress-strain curve has no linear range. One would expect, therefore, the subsequent exact results obtained, based on this strain energy function, to be meaningful when the experimental stress-strain curve is well approximated by the idealized one. This can happen, say, in the range of moderately large strains where the stress-strain curve is nonlinear or when the stress-strain curves for some materials have a small linear range.

The Cauchy stresses for this problem, near the crack-tip, are

$$\begin{Bmatrix}\sigma_{31}\\ \sigma_{32}\end{Bmatrix} \sim 2An\left\{\frac{Jh(\theta)}{\pi An}\right\}^{(1-(1/2n))} r^{-(1-1/2n)} \begin{Bmatrix}-\sin\phi\\ \cos\phi\end{Bmatrix} \text{ as } r \to 0$$

$$\sigma_{33} \sim \frac{2Jh(\theta)}{\pi} r^{-1} \tag{15}$$

where $h(\theta) = (\sin 2\phi/2\sin\theta)$, $2\phi = \theta + \arcsin(-[(n-1)/n]\sin\theta)$ and J is the J integral,† first found for full nonlinear elasticity by Eshelby[6]. From the exact solution, we find

$$J = \pi a(2An)\gamma_\infty^{2n}\left[-Q\left(\frac{1}{2n-1}\right)n\right] \tag{16}$$

where, from [2]

$$Q(n) = -\frac{n^{3/2}2^{1/\sqrt{n}}N_-(n,-1)\prod_{k=1}^{\infty}\left(\gamma_{2k}^{+} - \frac{1}{2k\sqrt{n}}\right)\exp(n+1)/4k\sqrt{n}}{(n+1)\exp(n+1)/2\sqrt{n}\prod_{k=1}^{\infty}\left(\gamma_{2k+1} - \frac{1}{(2k+1)\sqrt{n}}\right)\exp(n+1)/2\sqrt{n}(2k+1)}$$

$$\gamma_k^{\pm} = n^{1/2}\left\{1/n - 1 \pm \left[\left(1-\frac{1}{n}\right)^2 + 4k^2/n\right]^{1/2}\right\}\Big/2k$$

$$N_-(n,s) = 2^{-s\sqrt{(n)}}\frac{\prod_{k=1}^{\infty}(\gamma_{2k-1}^{+} - a_{2k-1}s)\exp(a_{2k-1}\bar{s})}{\prod_{k=1}^{\infty}(\gamma_{2k}^{+} - a_{2k}s)\exp(a_{2k}\bar{s})}$$

$$\bar{s} = s + (n-1)/2n, \qquad a_k = n^{1/2}/k. \tag{17}$$

$Q(n)$ is evaluated numerically using the above formulas. Numerical results are given in Table 1 to four significant figures.

When $n = 1$, the expression for $\sigma_{3\alpha}$ (see (15)) reduces to the well-known result involving the inverse square root r singularity. However, this is somewhat deceptive since the problem is

† $J = \int_\Gamma \left(W\,dx_2 - \tau_{3\beta}n_\beta \frac{\partial u_3}{\partial x_1}ds\right)$ where Γ is a simple contour in the $x_1 - x_2$ plane.

Table 1.

n	$Q\left(\frac{1}{2n-1}\right)$
2/3	−1.3674
1	−0.5
2	−0.1519
3	−0.0820
5	−0.0385

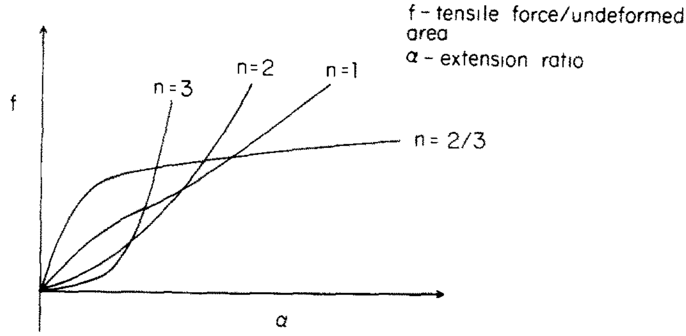

Fig. 2. Uniaxial tension curve.

inherently nonlinear even when $n = 1$, and the stress components must still be interpreted as belonging to the Cauchy stress tensor. More significantly, σ_{33}, like the strain energy density, always has an r^{-1} singularity independent of n and this component is completely absent in a small strain theory. Another interesting feature about σ_{33} is that it remains tensile throughout the body. Although the physical significance of the component σ_{33} for fracture can only be speculated upon, its tensile character and the nature of its singularity are peculiar nonetheless.

Figure 3 shows that for $(x_1/a) < (x_1/a)_{cr.}$, σ_{33} becomes the dominant stress component compared to σ_{32} ahead of the crack-tip. The behavior of σ_{32} in front of the crack-tip is shown in Fig. 4 for different values of the hardening (or softening) parameter n.

In the case where the strain energy density is a function of I_1 and I_2, it was noted by Adkins [4] that two additional non-trivial equilibrium equations appear and they combine to give a compatibility equation. If the strain energy function is of the form

$$W = A(I_1 - 3)^n + B(I_2 - 3)^n \tag{18}$$

where B is also a constant, the exact solution we have obtained still applies where now Σ is taken to be $2[(\partial W/\partial I_1) + (\partial W/\partial I_2)]$. This can be verified directly by noting that the compatibility

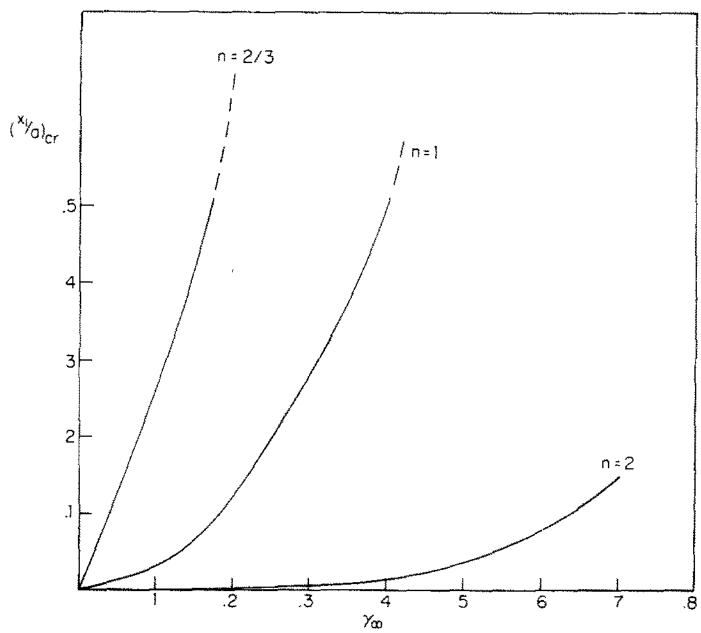

Fig. 3. For $(x_1/a) < (x_1/a)_{cr}$, σ_{33} becomes the dominant stress component compared to σ_{32} ahead of the crack-tip at $\theta = 0$.

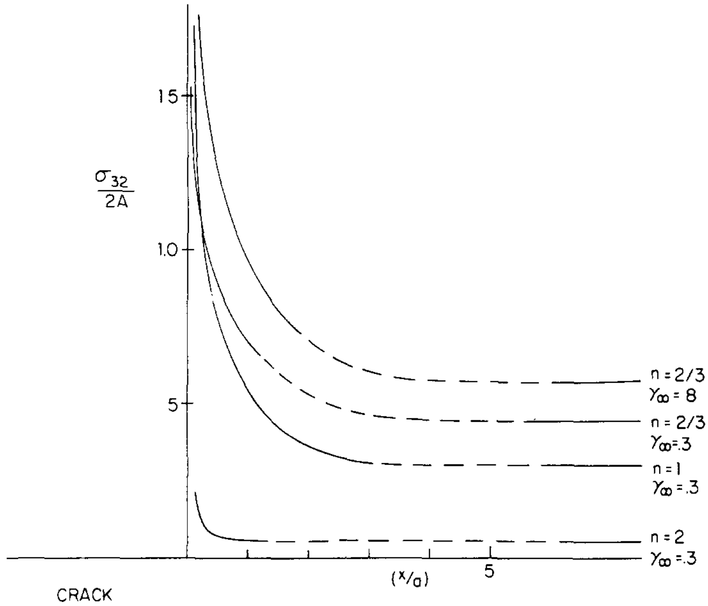

Fig. 4. Behavior of σ_{32} ahead of the crack-tip.

equation is satisfied identically if (18) is used. If the exponents in (18) are different, the exact solution no longer applies but the dominant singular behavior is determined by the higher of the two exponents. Finally, we note that solutions of all small strain anti-plane shear problems based on deformation theory of plasticity, as in [5], are also solutions of the corresponding finite-deformation anti-plane simple shear problems.

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